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A Physical Interpretation of the Maximum Likelihood Estimation of a Linear Functional Relationship Model

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Abstract: This paper presents a physical interpretation of estimating the parameters of a linear functional relationship model. The estimates are determined from the stable equilibrium position of a mechanical system. It is found that the estimates obtained from the physical model coincide with the maximum likelihood estimates.

1 Introduction

Consider a linear functional relationship $Y=\alpha+\beta X$, where X and Y are observed with errors u and v respectively, so that

$$y_i=\alpha+\beta(x_i-u_i)+v_i$$

with

$$y_i=Y_i+v_i$$

and

$$x_i=X_i+u_i, \qquad i=1, 2, \ldots, n \tag{1.1}$$

We assume that u and v have zero means and positive variances σ_u^2 and σ_v^2 respectively, and that they are both mutually uncorrelated. Estimations of this model using instrumental variables, multiple equations and repeated observations are proposed in the literature (e.g. see Maddala, 1977). The infeasibility of maximum likelihood estimation of this model was discussed in Lindley (1947) and Neyman and Scott (1951). However, the problem can be solved using the maximum likelihood method if the ratio of the error variances $\lambda=\sigma_v^2/\sigma_u^2$ is known.

In this paper, we present a mechanical model to interpret estimation of the parameters α and β in (1.1) with known λ. The model is natural and simple, and requires only elementary knowledge in mechanics.

2 The model

We first assume that $\lambda=1$ and set up the following physical model. Suppose that a weightless infinite-length rigid rod with infinitely small diameter is floating on the two-dimensional plane. The rod is free to move and rotate in any direction. n identical

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dimensionless particles are fixed at the points (x_i, y_i), $i=1, 2, \ldots, n$. We assume that each particle exerts a pulling force F_i orthogonal to the rod, and that the magnitude of F_i is proportional to the perpendicular distance between the particle and the rod. Without loss of generality, we assume that the constant of proportionality is 1. Hence, each particle tends to pull the rod close to itself and the force increases as the distance increases. When there is no net translatory force nor turning moment on the rod it is said to be in equilibrium. The location of the rod is then regarded as the best fitted equation.

Let

$$y = a + bx \tag{2.1}$$

(b is finite) be the position of the rod at an instant. The pulling force exerted on the rod from the particle at (x_i, y_i) is (see Figure 1)

$$F_i = (y_i - a - bx_i) \cos\theta \tag{2.2}$$

where $\theta = \tan^{-1} b$. A positive value of F_i indicates an upward orthogonal force on the rod and a negative value otherwise.

The first condition for the rod to be in equilibrium is that the sum of external forces exerted on the rod equals zero, i.e.

$$\sum_{i=1}^{n} F_i = 0 \tag{2.3}$$

or

$$\sum_{i=1}^{n} (y_i - a - bx_i) \cos\theta = 0. \tag{2.4}$$

From (2.4), we have

$$\bar{y} = a + b\bar{x} \tag{2.5}$$

where $\bar{x}$ and $\bar{y}$ are the sample means of $\{x_i\}$ and $\{y_i\}$ respectively. Hence, in equilibrium, the rod must pass through the centre of mass $(\bar{x}, \bar{y})$ of the particles, and the rod is only free to rotate about the pivot point $(\bar{x}, \bar{y})$.

The turning moment about $(\bar{x}, \bar{y})$ associated with F_i is

$$M_i = F_i\, d_i \tag{2.6}$$

where d_i is the displacement between $(\bar{x}, \bar{y})$ and the line of force of F_i (see Figure 2). A positive value of M_i indicates an anticlockwise turning moment about $(\bar{x}, \bar{y})$ and a negative value otherwise. The sum of turning moments about $(\bar{x}, \bar{y})$ is

$$\begin{aligned}\sum_{i=1}^{n} M_i &= \sum_{i=1}^{n} (y_i - a - bx_i) \cos\theta \{(x_i - \bar{x})/\cos\theta + (y_i - a - bx_i) \sin\theta\} \\ &= -\{b^2 S_{xy} - b(S_{yy} - S_{xx}) - S_{xy}\}/(1+b^2), \end{aligned}\tag{2.7}$$

where

$$S_{xx} = \sum_{i=1}^{n} (x_i - \bar{x})^2,\ S_{yy} = \sum_{i=1}^{n} (y_i - \bar{y})^2$$

and

$$S_{xy} = \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}).$$

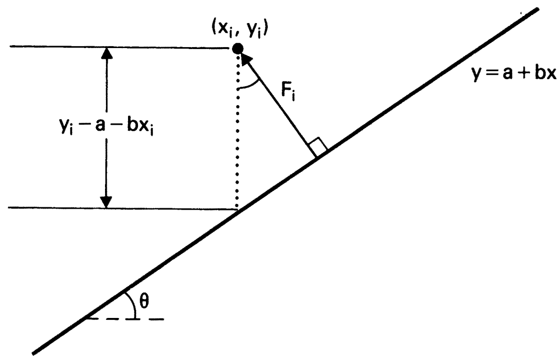

Fig. 1. Pulling force exerted on the rod from the particle at (x_i, y_i).

When the rod is in equilibrium,

$$\sum_{i=1}^{n} M_i = 0$$

We obtain

$$b^2 S_{xy} - b(S_{yy} - S_{xx}) - S_{xy} = 0. \tag{2.8}$$

Solving equation (2.8) for b, we have

$$b_1 = [S_{yy} - S_{xx} + \{(S_{yy} - S_{xx})^2 + 4S_{xy}{}^2\}^{1/2}]/(2S_{xy}) \tag{2.9}$$

and

$$b_2 = [S_{yy} - S_{xx} - \{(S_{yy} - S_{xx})^2 + 4S_{xy}{}^2\}^{1/2}]/(2S_{xy}). \tag{2.10}$$

Note that $(S_{yy} - S_{xx})^2 + 4S_{xy}{}^2 \geq 0$, therefore solutions always exist.

It can be shown that solutions (2.9) and (2.10) are the slopes of the rod when the rod is in stable and unstable equilibrium respectively. Hence solution (2.10) is discarded and the parameter estimates are given by

$$\hat{\beta} = b_1 \tag{2.11}$$

and

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}. \tag{2.12}$$

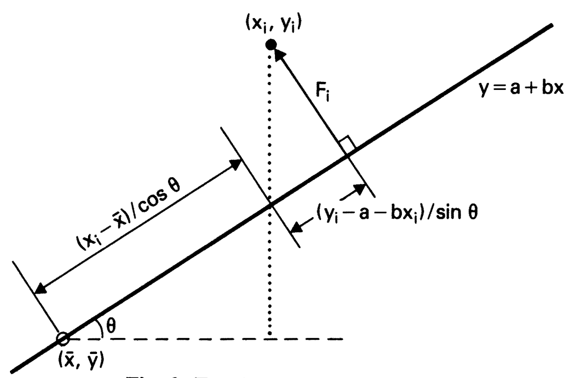

Fig. 2. Turning moment of force F_i.

For the case $\lambda = 1$, we let $z_i = \lambda^{1/2} x_i$, $w_i = \lambda^{1/2} u_i$ and $\beta^* = \lambda - \tfrac{1}{2}\beta$, so that (1.1) becomes

$$y_i = \alpha + \beta^* \ (z_i - w_i) + v_i \tag{2.13}$$

Here we have $\sigma_v^2/\sigma_w^2 = 1$ and the mechanical model applies for estimating β^*. It is then easy to show that

$$\hat{\beta} = [S_{yy} - \lambda S_{xx} + \{(S_{yy} - \lambda S_{xx})^2 + 4\lambda \ S_{xy}^2\}^{1/2}]/(2S_{xy}) \tag{2.14}$$

and

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} \tag{2.15}$$

3 Conclusions

We have presented a physical model to find the best fitted line of a linear functional relationship model. The parameter estimates are determined from the stable equilibrium state of the system and they happen to be the maximum likelihood estimates. This result also applies to the case that we know both σ_v^2 and σ_u^2, which was studied by Barnett (1967). In the case of simple linear regression model, we can assume that each particle exerts a vertical pulling force on the rod and this has been studied by Hui (1983).

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